

Décalage and Kan's simplicial loop group functor

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Abstract

Given a bisimplicial set, there are two ways to extract from it a simplicial set: the diagonal simplicial set and the less well known total simplicial set of Artin and Mazur. There is a natural comparison map between these simplicial sets, and it is a theorem due to Cegarra and Remedios and independently Joyal and Tierney, that this comparison map is a weak equivalence for any bisimplicial set. In this paper we will give a new, elementary proof of this result. As an application, we will revisit Kan's simplicial loop group functor G . We will give a simple formula for this functor, which is based on a factorization, due to Duskin, of Eilenberg and Mac Lane's classifying complex functor \overline{W} . We will give a new, short, proof of Kan's result that the unit map for the adjunction $G \dashv \overline{W}$ is a weak equivalence for reduced simplicial sets.

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1 Introduction

The aim of this paper is to give new and hopefully simpler proofs of two theorems in the theory of simplicial sets, the first being a generalization to simplicial sets of Dold-Puppe's version [6] of the Eilenberg-Zilber theorem from homological algebra, the second being an old result of Kan's on simplicial loop groups. This first result is due to Cegarra and Remedios and independently to Joyal and Tierney.

Recall that if C is a double complex of abelian groups concentrated in the first quadrant then there are two ways in which one can associate to it an ordinary complex. One can form the total complex $\text{Tot } C$ which in degree n is equal to

$$(\text{Tot } C)_n = \bigoplus_{p+q=n} C_{p,q},$$

or one can form the diagonal complex dC which in degree n is equal to

$$(dC)_n = C_{n,n}.$$

There is a natural comparison map $dC \rightarrow \text{Tot } C$ and the generalized Eilenberg-Zilber theorem [6, 11] says that this comparison map is a chain homotopy equivalence.

There is a generalization of this comparison with chain complexes, or equivalently simplicial abelian groups, replaced by simplicial sets. Just as we can form the diagonal of a double complex we can also form the diagonal dX of a *bisimplicial set* X . This is the simplicial set obtained by precomposing the functor $X: \Delta^{\text{op}} \times \Delta^{\text{op}} \rightarrow \mathbf{Set}$ with the opposite of the functor $\delta: \Delta \rightarrow \Delta \times \Delta$ given by $\delta([n]) = ([n], [n])$, i.e. $dX = X\delta$, so that the set of n -simplices of dX is $(dX)_n = X_{n,n}$. There is another, less well known, way to form a simplicial set from X . Namely one can form what is variously known as the *total simplicial set* or *Artin-Mazur codiagonal* TX of X (see [1]). This construction extends to define a functor $T: \mathbf{SS} \rightarrow \mathbf{S}$ from the category \mathbf{SS} of bisimplicial sets to the category \mathbf{S} of simplicial sets.

The construction TX is the analog for simplicial sets of the process of forming the total complex $\text{Tot } C$ of a double complex C . In fact, if

$$N: s\mathbf{Ab} \rightleftharpoons \mathbf{Ch}_{\geq 0}: \Gamma$$

denotes the Dold-Kan correspondence, then NTA is isomorphic to $\text{Tot } NA$ (see [3]).

As mentioned above, the Eilenberg-Zilber theorem in homological algebra has a generalization for simplicial sets. For any bisimplicial set X , there is a natural comparison map

$$dX \rightarrow TX$$

between the diagonal simplicial set of X and the total simplicial set of X . We have the following result.

Theorem 1 ([3], [17]). *Let X be a bisimplicial set. Then the comparison map $dX \rightarrow TX$ is a weak equivalence.*

The first published proof of this result was given in [3], with the authors noting that this fact is stated without proof in [5] where it is attributed to Zisman (unpublished). When X is a bisimplicial group, a closely related result was proven by Quillen [22]. The proof of Theorem 1 given in [3] is unfortunately somewhat complicated, and so it is of interest to have a simpler proof. One such proof, incorporating some ideas of Cisinski, is given by Joyal and Tierney in their forthcoming book [17]. We shall give here a new proof, which we think is fairly elementary — in particular it uses nothing more than the fact that the diagonal functor d sends level-wise weak equivalences to weak equivalences.

In the second part of the paper we present a simple construction of Kan's simplicial loop group functor as an application of Theorem 1. Recall that in [13], Kan defined a functor $G: \mathbf{S} \rightarrow s\mathbf{Gp}$ which is left adjoint to the classifying complex functor $\overline{W}: s\mathbf{Gp} \rightarrow \mathbf{S}$ of Eilenberg and Mac Lane [10]. He was able to show that, when X is reduced (i.e. when X is a simplicial set with only one vertex), the principal GX bundle X_η on X induced by the unit map $\eta: X \rightarrow \overline{W}GX$ has weakly contractible total space. Kan's proof of this last fact involves showing firstly that X_η is simply connected, and secondly that X_η is acyclic in the sense that it has vanishing reduced homology in all degrees.

We will show that both the construction of the functor G and the proof that the unit map is a weak equivalence can be greatly illuminated and simplified by considering a factorization (first noticed by Duskin) of \overline{W} involving the functor T . In fact we hasten to point out that this last section of the paper makes no great claim to originality, we find it hard to believe that some of the results of this section were not known to Duskin, although we cannot find any evidence for this in his published papers. We also point out that in their forthcoming book [17] and their paper [18] Joyal and Tierney prove more general statements in the context of simplicially enriched groupoids. Using Duskin's factorization we will give a simple formula for the left adjoint to \overline{W} (see Proposition 16). In Theorem 21 we will apply this formula to give a simple and direct proof of Kan's theorem that the unit map of the adjunction $G \dashv \overline{W}$ is a weak equivalence whenever X is reduced. To the best of our knowledge this proof is new (we note that an essential ingredient for the proof is Theorem 1). We point out that in [24] Waldhausen described another approach to the construction of G , nevertheless we feel our approach (which proceeds along different lines) is still of some interest.

2 The décalage comonad

We begin by recalling the definition and main properties of the décalage and total décalage functors of Illusie [14].

2.1 The décalage or shift functor

Let Δ_a denote the augmented simplex category, in other words the simplex category Δ together with the additional object $[-1]$, the empty set (the initial object of Δ_a). We will write $as\mathcal{C}$ for the category $[\Delta_a^{\text{op}}, \mathcal{C}]$ of augmented simplicial objects in a category \mathcal{C} , which we will assume to be complete and cocomplete. Recall (see for example VII.5 of [19]) that Δ_a is a monoidal category with unit $[-1]$ under the operation of ordinal sum, which operation we will denote by σ (following Joyal and Tierney). If $[m], [n] \in \Delta_a$ then $\sigma([m], [n]) = [m+n+1]$, and the operation σ gives rise to a bifunctor $\sigma: \Delta_a \times \Delta_a \rightarrow \Delta_a$ which sends a morphism

$$(\alpha, \beta): ([m], [n]) \rightarrow ([m'], [n'])$$

in $\Delta_a \times \Delta_a$ to the morphism $\sigma(\alpha, \beta): [m+n+1] \rightarrow [m'+n'+1]$ in Δ_a defined by

$$\sigma(\alpha, \beta)(i) = \begin{cases} \alpha(i) & \text{if } 0 \leq i \leq m \\ \beta(i - m - 1) + m' + 1 & \text{if } m + 1 \leq i \leq m + n + 1. \end{cases}$$

(Δ_a, σ) is not a symmetric monoidal category — while $\sigma([m], [n]) = \sigma([n], [m])$, it need not be the case that $\sigma(\alpha, \beta) = \sigma(\beta, \alpha)$. The monoidal structure on Δ_a allows us to define a functor $\sigma(-, [0]): \Delta_a \rightarrow \Delta$ which sends $[n] \in \Delta_a$ to $\sigma([n], [0]) = [n+1]$ in Δ . We have the following definition which we believe is originally due to Illusie.

Definition 2 ([14]). Define $\text{Dec}_0: s\mathcal{C} \rightarrow as\mathcal{C}$ to be the functor given by restriction along $\sigma(-, [0]): \Delta_a \rightarrow \Delta$, so that if X is a simplicial object in \mathcal{C} then $\text{Dec}_0 X$ is the augmented simplicial object given by

$$\text{Dec}_0 X([n]) = X([n+1]),$$

whose face maps $d_i: (\text{Dec}_0 X)_n \rightarrow (\text{Dec}_0 X)_{n-1}$ are given by $d_i: X_{n+1} \rightarrow X_n$ for $i = 0, 1, \dots, n$, and whose degeneracy maps $s_i: (\text{Dec}_0 X)_n \rightarrow (\text{Dec}_0 X)_{n+1}$ are given by $s_i: X_{n+1} \rightarrow X_{n+2}$ for $i = 0, 1, \dots, n$. The augmentation $(\text{Dec}_0 X)_0 \rightarrow X_0$ is given by $d_0: X_1 \rightarrow X_0$.

$\text{Dec}_0 X$ is obtained from X by forgetting the top face and degeneracy map at each level and re-indexing by shifting degrees up by one. Thus the augmented simplicial object $\text{Dec}_0 X$ can be pictured as

$$\begin{array}{ccccccc} & & & \xleftarrow{d_0} & & & \\ & & & \xleftarrow{d_1} & & & \\ & & & \xleftarrow{d_2} & & & \\ X_0 & \xleftarrow{d_0} & X_1 & \xleftarrow{d_1} & X_2 & \xleftarrow{d_2} & X_3 & \dots \\ & & \xrightarrow{s_0} & & \xrightarrow{s_0} & & \xrightarrow{s_1} & \\ & & & & & & & \end{array}$$

Note that the simplicial identity $d_0 d_1 = d_0 d_0$ shows that $d_0: X_1 \rightarrow X_0$ is an augmentation.

There is an analogous functor $\text{Dec}^0: s\mathcal{C} \rightarrow as\mathcal{C}$ given by restriction along the functor $\sigma([0], -): \Delta_a \rightarrow \Delta$ — thus Dec^0 is the functor which forgets the bottom face and degeneracy map at each level. The functors Dec_0 and Dec^0 are usually called the *décalage* or *shifting* functors. More generally we can define functors $\text{Dec}_n: s\mathcal{C} \rightarrow as\mathcal{C}$ and $\text{Dec}^n: s\mathcal{C} \rightarrow as\mathcal{C}$ induced by restriction along $\sigma(-, [n]): \Delta_a \rightarrow \Delta$ and $\sigma([n], -): \Delta_a \rightarrow \Delta$ respectively.

The relation between $\text{Dec}_n X$ and $\text{Dec}^n X$ can be easily understood through the notion of the opposite simplicial object. Let $\tau: \Delta \rightarrow \Delta$ denote the automorphism of Δ which reverses the order of each ordinal $[n]$, or equivalently sends the category $[n]$ to its opposite category. Note that $\tau(\sigma([n], [m])) = \sigma(\tau([m]), \tau([n]))$ for any $[n], [m] \in \Delta$. If X is a simplicial object then we write X^o for the simplicial object obtained by precomposing X with the functor τ^{op} . The simplicial object X^o is called the *opposite* simplicial object of X in [15]. Note that $(\text{Dec}_0 X)^o = \text{Dec}^0(X^o)$ by the following calculation:

$$(\text{Dec}_0 X)^o([n]) = \text{Dec}_0 X(\tau([n]) = X(\sigma([0], \tau([n]))) = X(\sigma(\tau([n]), [0])),$$

since $\tau[0] = [0]$. It follows that $(\text{Dec}_n X)^o = \text{Dec}^n(X^o)$ for any $n \geq 0$.

There are canonical comonads underlying the functors Dec_0 and Dec^0 , when these functors are thought of as endofunctors on $s\mathcal{C}$ by forgetting augmentations. As is well known, $[0]$ determines a monoid in Δ whose multiplication is given by the canonical map $[1] \rightarrow [0]$. This monoid is universal in a certain precise sense (see Proposition 5.1 in Chapter VII of [19]).

The monoid $[0]$ determines a corresponding comonoid in Δ^{op} which in turn induces by composition the two comonads Dec_0 and Dec^0 in $s\mathcal{C}$. The counit of the comonad Dec_0 is induced by the natural transformation $[n] \rightarrow \sigma([0], [n])$ and hence is given on a simplicial object X by the simplicial map $\text{Dec}_0 X \rightarrow X$ which in degree n is the last face map $d_{n+1}: X_{n+1} \rightarrow X_n$. We will write $d_{\text{last}}: \text{Dec}_0 X \rightarrow X$ for this map.

Likewise, the counit of the comonad Dec^0 is induced by the natural transformation $[n] \rightarrow \sigma([n], [0])$ and hence is given on a simplicial object X by the simplicial map $\text{Dec}^0 X \rightarrow X$ which in degree n is the first face map $d_0: X_{n+1} \rightarrow X_n$. We will write $d_{\text{first}}: \text{Dec}^0 X \rightarrow X$ for this map.

When Dec_0 and Dec^0 are regarded as endofunctors on $s\mathcal{C}$, we see that the functors Dec_n and Dec^n (also thought of as endofunctors on $s\mathcal{C}$) are given by $\text{Dec}_n = (\text{Dec}_0)^n$ and $\text{Dec}^n = (\text{Dec}^0)^n$ respectively.

2.2 Contractibility of the décalage functor

It is an important fact that $\text{Dec}_0 X$ and $\text{Dec}^0 X$ are not just augmented simplicial objects, they are actually contractible augmented simplicial objects in the following sense.

Recall that the augmentation map of an augmented simplicial object $\epsilon: X \rightarrow X_{-1}$ is a deformation retraction if there exists a simplicial map $s: X_{-1} \rightarrow X$ (with X_{-1} is regarded as a constant simplicial object) which is a section of the projection ϵ and is such that $s\epsilon$ is simplicially homotopic to the identity map on X .

A sufficient condition for $s\epsilon$ to be simplicially homotopic to the identity map on X is that there exist for each $n \geq -1$, maps $s_{n+1}: X_n \rightarrow X_{n+1}$ with $s_0 = s$, which act as ‘extra degeneracies on the right’ in the sense that the following identities hold:

$$\begin{aligned}
d_i s_n &= s_{n-1} d_i \text{ for } 0 \leq i < n, \\
d_n s_n &= \text{id}, \\
s_i s_n &= s_{n+1} s_i \text{ for } 0 \leq i \leq n,
\end{aligned}$$

The following definition is standard.

Definition 3. Let $\epsilon: X \rightarrow X_{-1}$ be an augmented simplicial object in \mathcal{C} . By a *contraction* of X we will mean the data of the section $s: X_{-1} \rightarrow X$ together with the extra degeneracies s_{n+1} as described above. We will say that X is *contractible* if it has such a contraction.

A map of contractible augmented simplicial objects is a map of the underlying augmented simplicial objects which preserves the corresponding sections s and the extra degeneracies (as in [7] we will sometimes say that such a map is *coherent*). We will write $a_c \mathcal{C}$ for the category of contractible augmented simplicial objects and coherent maps.

Given the data of such a collection of maps s_{n+1} as above, we define maps $h_i: X_n \rightarrow X_{n+1}$ by the formula

$$h_i = s_0^{n-i} s_{n+1} d_0^{n-i}.$$

It is easy to check that the maps h_i satisfy the conditions (i)–(iii) in Definition 5.1 of [20]. The h_i then piece together to define a simplicial homotopy $h: X \otimes \Delta[1] \rightarrow X$ from $s\epsilon$ to the identity on X , analogous to Proposition 6.2 in [20]. Here, if K is a simplicial set, $X \otimes K$ denotes the tensor for the usual structure of $s\mathcal{C}$ as a simplicially enriched category, so that $X \otimes K$ has n -simplices given by

$$(X \otimes K)_n = \coprod_{k \in K_n} X_n. \tag{1}$$

In degree n , the map $h: X \otimes \Delta[1] \rightarrow X$ is given by $d_{i+1} h_i: (X_n)_\alpha \rightarrow X_n$ on the summand $(X_n)_\alpha$ of $(X \otimes \Delta[1])_n$ corresponding to the map $\alpha: [n] \rightarrow [1]$ determined by $\alpha^{-1}(0) = [i]$. We summarize this discussion in the following lemma.

Lemma 4. *Let $\epsilon: X \rightarrow X_{-1}$ be a contractible augmented simplicial object in \mathcal{C} . Then there is a simplicial homotopy $h: X \otimes \Delta[1] \rightarrow X$ in $s\mathcal{C}$ between $s\epsilon$ and 1_X .*

Clearly, the degeneracy $s_{n+1}: X_{n+1} \rightarrow X_{n+2}$ for $n \geq 0$ equips $\text{Dec}_0 X$ with an extra degeneracy in the above sense. Therefore we have the following well known result.

Lemma 5. *For any simplicial object X in \mathcal{C} , the augmentation $d_0: \text{Dec}_0 X \rightarrow X_0$ is a deformation retract. An analogous statement is true for $\text{Dec}^0 X$.*

A prime example where simplicial objects with extra degeneracies appear is in the construction of simplicial comonadic resolutions. Suppose that L is a comonad on a category \mathcal{C} , and X is an object of \mathcal{C} . Then, as is well known, L determines an augmented simplicial

object L_*X whose object of n -simplices is L^nX and whose face and degeneracy maps are defined by

$$d_i = L^i \epsilon L^{n-i}, s_i = L^i \delta L^{n-i-1}$$

respectively, where $\epsilon: L \rightarrow 1$ denotes the counit and $\delta: L \rightarrow L^2$ denotes the comultiplication of the comonad. Suppose that there exists a section $\sigma: X \rightarrow LX$ of the counit $\epsilon_X: LX \rightarrow X$. Then σ determines extra degeneracies $s_{n+1}: L^nX \rightarrow L^{n+1}X$ given by $s_{n+1} = L^n\sigma$ (see for example [25]). It follows from the discussion above that there is a simplicial homotopy $h: L_*X \otimes \Delta[1] \rightarrow L_*X$ in $s\mathcal{C}$ between $\sigma\epsilon$ and the identity on L_*X .

3 The total décalage and the total simplicial set functors

In this section we will recall some of the main properties of Illusie's total décalage functor Dec [14] and its right adjoint, the Artin-Mazur total simplicial set functor [1]. For more details the reader should refer to the excellent discussion of these functors and their properties in the papers [3, 4]. In this section we will mainly be interested in the case where $\mathcal{C} = \mathbf{Set}$. We begin therefore by explaining our notations and conventions for bisimplicial sets (which follows closely the presentation in [16]).

If $X \in \mathbf{SS}$ is a bisimplicial set then we will say that $X_{m,n} = X([m], [n])$ has horizontal degree m and vertical degree n . We write \mathbf{SS} for the category of bisimplicial sets. We say a *simplicial space* is a simplicial object in \mathbf{S} . There are two ways in which we can regard a bisimplicial set X as a simplicial space. On the one hand, we can define X_m to be the simplicial set with n -simplices $(X_m)_n = X_{m,n}$. Thus we regard X as a horizontal simplicial object with vertical simplicial sets. On the other hand we can define X_n to be the simplicial set with m -simplices $(X_n)_m = X_{m,n}$. Thus we regard X as a vertical simplicial object with horizontal simplicial sets.

Each of these two ways of viewing a bisimplicial set as a simplicial space leads to a simplicial enrichment of \mathbf{SS} , using the canonical simplicial enrichment of $s\mathbf{S}$ mentioned earlier. If we view bisimplicial sets as a horizontal simplicial objects in \mathbf{S} , then $\mathbf{SS} = s\mathbf{S}$ is equipped with the structure of a simplicial enriched category for which the tensor $X \otimes_1 K$, for X a bisimplicial set and $K \in \mathbf{S}$, has vertical simplicial set of m -simplices given by (see (1))

$$(X \otimes_1 K)_m = \coprod_{k \in K_m} X_m = X_m \times K_m,$$

so that the set of (m, n) -bisimplices of $X \otimes_1 K$ is $(X \otimes_1 K)_{m,n} = X_{m,n} \times K_m$. In other words,

$$X \otimes_1 K = X \times p_1^* K,$$

where $p_1: \Delta \times \Delta \rightarrow \Delta$ denotes projection onto the first factor. The simplicial enrichment is then defined by the formula

$$\text{Hom}_1(X, Y) = i_1^*(Y^X),$$

where $i_1: \Delta \rightarrow \Delta \times \Delta$ denotes the right adjoint to p_1 , so that $i_1([n]) = ([n], [0])$.

Similarly, if we view $X \in \mathbf{SS}$ as a vertical simplicial object, then the tensor $X \otimes_2 K$ is given by

$$X \otimes_2 K = X \times p_2^* K$$

where $p_2: \Delta \times \Delta \rightarrow \Delta$ denotes projection onto the second factor. The simplicial enrichment is defined by the formula

$$\mathrm{Hom}_2(X, Y) = i_2^*(Y^X),$$

where $i_2: \Delta \rightarrow \Delta \times \Delta$ denotes right adjoint to p_2 , defined by $i_2([n]) = ([0], [n])$.

Following Joyal we will say that a bisimplicial set X is *row augmented* if there is a map $X \rightarrow p_1^* K$ in \mathbf{SS} for some simplicial set K , and we will say that X is *column augmented* if there is a map $X \rightarrow p_2^* K$ in \mathbf{SS} for some simplicial set K .

With these conventions understood we can describe Illusie's total décalage functor [14]. The simplicial comonadic resolution of Dec_0 gives rise to a functor $\mathrm{Dec}: \mathbf{S} \rightarrow s\mathbf{S}$ which sends a simplicial set X to the simplicial space $\mathrm{Dec}X$ which in degree n is the simplicial set

$$\mathrm{Dec}_n X = (\mathrm{Dec}_0)^n X.$$

Here we are thinking of $\mathrm{Dec}X$ as a vertical simplicial object in \mathbf{S} with horizontal simplicial sets. The set of (m, n) -bisimplices of the bisimplicial set $\mathrm{Dec}X$ is $(\mathrm{Dec}X)_{m,n} = X_{m+n+1}$. The horizontal and vertical face operators $d_i^h: (\mathrm{Dec}X)_{m+1,n} \rightarrow (\mathrm{Dec}X)_{m,n}$ and $d_i^v: (\mathrm{Dec}X)_{m,n+1} \rightarrow (\mathrm{Dec}X)_{m,n}$ are given by $d_i^h = d_i: X_{m+n+2} \rightarrow X_{m+n+1}$ and $d_i^v = d_{m+i+1}: X_{m+n+2} \rightarrow X_{m+n+1}$ respectively. There are similar formulas for the horizontal and vertical degeneracy operators. Note that if we regard the bisimplicial set $\mathrm{Dec}X$ as a horizontal simplicial object with vertical simplicial sets then $\mathrm{Dec}X$ is the simplicial comonadic resolution of X by Dec^0 . The following Lemma is straightforward.

Lemma 6. *The functor $\mathrm{Dec}: \mathbf{S} \rightarrow \mathbf{SS}$ is given by restriction along the ordinal sum map $\sigma: \Delta \times \Delta \rightarrow \Delta$ so that*

$$\mathrm{Dec} X([m], [n]) = X(\sigma([m], [n])) = X_{n+m+1}$$

for $X \in \mathbf{S}$.

We see from this Lemma that in fact $\mathrm{Dec}X$ is the restriction of a bi-augmented simplicial object in the sense that the functor $\mathrm{Dec}: \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \rightarrow \mathcal{C}$ extends canonically to a functor $(\Delta_a^{\mathrm{op}} \times \Delta^{\mathrm{op}}) \cup (\Delta^{\mathrm{op}} \times \Delta_a^{\mathrm{op}}) \rightarrow \mathcal{C}$. Therefore $\mathrm{Dec}X$ is both row and column augmented. The row augmentation $\epsilon_r: \mathrm{Dec}X \rightarrow p_1^* X$ is given by the map $d_{\mathrm{last}}: \mathrm{Dec}_0 X \rightarrow X$, while the column augmentation $\epsilon_c: \mathrm{Dec}X \rightarrow p_2^* X$ is given by the map $d_{\mathrm{first}}: \mathrm{Dec}^0 X \rightarrow X$.

Suppose that X is a simplicial set, and regard $\mathrm{Dec}X$ as a (vertical) simplicial space whose rows are the simplicial sets $\mathrm{Dec}_n X$ for $n \geq 0$. Then the functor $\mathrm{disc}: \mathbf{S} \rightarrow \mathbf{SS}$ which sends a simplicial set K to the constant simplicial space whose rows are K , has a left adjoint $\pi_0: \mathbf{SS} \rightarrow \mathbf{S}$. Note that the functor disc is nothing other than the functor $p_1^*: \mathbf{S} \rightarrow \mathbf{SS}$. It is easy to compute the value of $\pi_0 \mathrm{Dec}X$ as the next lemma shows.

Lemma 7. *For any simplicial set X , we have $\pi_0 \text{Dec} X = X$.*

Proof. The value of the functor $\pi_0: \mathbf{SS} \rightarrow \mathbf{S}$ on a bisimplicial set Y is given by the coequalizer

$$\pi_0 Y = \text{coeq}(Y_1 \xrightleftharpoons[d_1]{d_0} Y_0),$$

where we have written Y_n for the n -th row of Y . Since colimits in \mathbf{S} are computed pointwise we see that the set of n -simplices $\pi_0(Y)_n$ is the set of path components of the n -th column of Y . Therefore the set of n -simplices of $\pi_0 \text{Dec} X$ is the set of path components of $\text{Dec}^n X$, which we have seen is equal to X_n . One can further check that this isomorphism is compatible with face and degeneracy maps so that we get the identification $\pi_0 \text{Dec} X = X$. \square

Thus when $X = \Delta[n]$ we have an isomorphism $\pi_0 \text{Dec} \Delta[n] = \Delta[n]$. In fact, we will show that more is true: we will see that the row and column augmentation maps $\text{Dec} \Delta[n] \rightarrow p_1^* \Delta[n]$ and $\text{Dec} \Delta[n] \rightarrow p_2^* \Delta[n]$ are simplicial homotopy equivalences. To see this we will need the following result.

Lemma 8. *For any $n \geq 0$ there are sections of the maps $d_{\text{last}}: \text{Dec}_0 \Delta[n] \rightarrow \Delta[n]$ and $d_{\text{first}}: \text{Dec}^0 \Delta[n] \rightarrow \Delta[n]$.*

Proof. Consider the map $\sigma_r: \Delta[n] \rightarrow \text{Dec}_0 \Delta[n]$ given in degree m by

$$\sigma_r(x) = s^{n+1} \sigma(x, [0])$$

for $x: [m] \rightarrow [n]$. Clearly this is natural in x . It is also a section of d_{last} by the following calculation. Since $d_{\text{last}} \sigma_r(x) = d_{m+1} s^{n+1} \sigma(x, [0]) = s^{n+1} \sigma(x, [0]) d^{m+1}$, then for any $i \in [m]$ we have

$$\begin{aligned} s^{n+1} \sigma(x, [0]) d^{m+1}(i) &= s^{n+1} \sigma(x, [0])(i) \\ &= s^{n+1}(x(i)) \\ &= x(i), \end{aligned}$$

so that $d_{\text{last}} \sigma_r = \sigma_r$. In a completely analogous way one can define a section of d_{first} . \square

The section $\sigma_r: \Delta[n] \rightarrow \text{Dec}_0 \Delta[n]$ is a section of the counit $d_{\text{last}}: \text{Dec}_0 \Delta[n] \rightarrow \Delta[n]$ and hence defines an extra degeneracy of the simplicial comonadic resolution $\text{Dec} \Delta[n]$ of $\Delta[n]$. As discussed in Section 2.2 above, this means that the map $\sigma_r: \Delta[n] \rightarrow \text{Dec} \Delta[n]$ exhibits $\epsilon_r: \text{Dec} \Delta[n] \rightarrow p_1^* \Delta[n]$ as a deformation retraction: thus $\epsilon_r \sigma_r$ is the identity on $p_1^* \Delta[n]$ and there is a simplicial homotopy between $\sigma_r \epsilon_r$ and the identity on $\text{Dec} \Delta[n]$. Here we are viewing $\text{Dec} \Delta[n]$ as a vertical simplicial object with horizontal simplicial sets and thus the simplicial homotopy is a map $h: \text{Dec} \Delta[n] \otimes_2 \Delta[1] \rightarrow \text{Dec} \Delta[n]$. Completely analogous statements apply for the section σ_c . We summarize this discussion in the following lemma.

Lemma 9. *There are maps $\sigma_r: p_1^* \Delta[n] \rightarrow \text{Dec} \Delta[n]$ and $\sigma_c: p_2^* \Delta[n] \rightarrow \text{Dec} \Delta[n]$ in \mathbf{SS} such that the following are true:*

1. σ_r is a section of $\epsilon_r: \text{Dec } \Delta[n] \rightarrow p_1^* \Delta[n]$ and σ_c is a section of $\epsilon_c: \text{Dec } \Delta[n] \rightarrow p_2^* \Delta[n]$,
2. there is a simplicial homotopy $h: \text{Dec } \Delta[n] \otimes_2 \Delta[1] \rightarrow \text{Dec } \Delta[n]$ from $\sigma_r \epsilon_r$ to the identity,
3. there is a simplicial homotopy $k: \text{Dec } \Delta[n] \otimes_1 \Delta[1] \rightarrow \text{Dec } \Delta[n]$ from $\sigma_c \epsilon_c$ to the identity.

From the description of Dec in Lemma 6 above it is clear that Dec has both a left and right adjoint. The left adjoint of Dec is related to the notion of the join of simplicial sets. The right adjoint to Dec is denoted $T: \mathbf{SS} \rightarrow \mathbf{S}$, it was introduced in [1] where it was called the *total simplicial set functor*. It is also known as the *Artin-Mazur codiagonal*. It has the following explicit description: if X is a bisimplicial set then the set $(TX)_n$ of n -simplices of the simplicial set TX is given by the equalizer of the diagram

$$(TX)_n \rightarrow \prod_{i=0}^n X_{i,n-i} \rightrightarrows \prod_{i=0}^{n-1} X_{i,n-i-1} \quad (2)$$

where the components of the two maps are defined by the composites

$$\prod_{i=0}^n X_{i,n-i} \xrightarrow{p_i} X_{i,n-i} \xrightarrow{d_0^v} X_{i,n-i-1}$$

and

$$\prod_{i=0}^n X_{i,n-i} \xrightarrow{p_{i+1}} X_{i+1,n-i-1} \xrightarrow{d_{i+1}^h} X_{i,n-i-1}.$$

The face maps $d_i: (TX)_n \rightarrow (TX)_{n-1}$ are given by

$$d_i = (d_i^v p_0, d_{i-1}^v p_1, \dots, d_1^v p_{i-1}, d_i^h p_{i+1}, d_i^h p_{i+2}, \dots, d_i^h p_n)$$

while the degeneracy maps $s_i: (TX)_n \rightarrow (TX)_{n+1}$ are given by

$$s_i = (s_i^v p_0, s_{i-1}^v p_1, \dots, s_0^v p_i, s_i^h p_{i+1}, \dots, s_i^h p_n).$$

The unit map $\eta: X \rightarrow T\text{Dec } X$ of the adjunction $\text{Dec} \dashv T$ is given by the map

$$x \mapsto (s_0(x), s_1(x), \dots, s_n(x)) \quad (3)$$

in degree n (see [3]). In general it is rather difficult to give a simple description of the simplicial set TX for an arbitrary bisimplicial set X . When X is constant however, we have the following well-known result.

Lemma 10. *Let X be a simplicial set. Then there are isomorphisms $Tp_1^* X = Tp_2^* X = X$, natural in X .*

Proof. Observe that the functor Tp_1^* is right adjoint to the functor $\pi_0 \text{Dec}$. Lemma 7 implies that the functor $\pi_0 \text{Dec}$ is the identity on \mathbf{S} , from which it follows that there is an isomorphism $Tp_1^* X = X$, natural in X . The other statement is proven in an analogous fashion. \square

4 The generalized Eilenberg-Zilber theorem for simplicial sets

Our goal in this section is to present an elementary proof of Theorem 1. Recall that this theorem states that there is a weak equivalence

$$dX \rightarrow TX \tag{4}$$

of simplicial sets, natural in X . As mentioned earlier, the proof of Theorem 1 in [3] is rather lengthy, and so it is of interest to have a simpler approach. We will describe here another proof, which we think is fairly elementary (as mentioned in the Introduction, the forthcoming book [17] of Joyal and Tierney contains another proof, which proceeds along different lines).

We begin by describing the map (4). This map is obtained from the map of cosimplicial bisimplicial sets

$$\text{Dec}\Delta \rightarrow (p_1^*\Delta \times p_2^*\Delta)\delta \tag{5}$$

by applying the functor $\mathbf{SS}(-, X): c\mathbf{SS} \rightarrow \mathbf{S}$. Here $(p_1^*\Delta \times p_2^*\Delta)\delta$ is the cosimplicial bisimplicial set which in degree n is $p_1^*\Delta[n] \times p_2^*\Delta[n]$. The map (5) in degree n is the canonical map induced by the row and column augmentations $\epsilon_r: \text{Dec}\Delta[n] \rightarrow p_1^*\Delta[n]$ and $\epsilon_c: \text{Dec}\Delta[n] \rightarrow p_2^*\Delta[n]$ respectively. Note that it is possible to describe the map $dX \rightarrow TX$ much more explicitly at the level of simplices (see [3]) but we will not need this.

The proof of Theorem 1 that we shall give essentially boils down to the well known fact that the diagonal functor $d: \mathbf{SS} \rightarrow \mathbf{S}$ sends level-wise weak equivalences of bisimplicial sets to weak equivalences of simplicial sets. In other words, if $f: X \rightarrow Y$ is a map in \mathbf{SS} such that the map $f_n: X_n \rightarrow Y_n$ on n -th rows is a weak equivalence for all $n \geq 0$, then $df: dX \rightarrow dY$ is also a weak equivalence. Alternatively, if the map $f_n: X_n \rightarrow Y_n$ on the n -th columns is a weak equivalence for all $n \geq 0$, then $df: dX \rightarrow dY$ is a weak equivalence.

Recall that d has a right adjoint $d_*: \mathbf{S} \rightarrow \mathbf{SS}$ (see for instance [11] page 222) defined by the formula

$$(d_*X)_{m,n} = \mathbf{S}(\Delta[m] \times \Delta[n], X). \tag{6}$$

Using the fact that the diagonal d sends level-wise weak equivalences to weak equivalences one can prove (see for instance [21]) that the counit $\epsilon: dd_*K \rightarrow K$ of this adjunction is a weak equivalence for any simplicial set K , and so in particular $dd_*TX \rightarrow TX$ is a weak equivalence for any bisimplicial set X . Therefore, since we can factor (4) as

$$dX \rightarrow dd_*TX \rightarrow TX,$$

we see that to prove Theorem 1 it suffices to prove the following proposition.

Proposition 11. *The map $dX \rightarrow dd_*TX$ is a weak equivalence for any bisimplicial set X .*

Proof. First note that the underlying map $X \rightarrow d_*TX$ of bisimplicial sets is induced by the following map in $cc\mathbf{SS}$:

$$\epsilon_r \times \epsilon_c: \text{Dec}\Delta \times \text{Dec}\Delta \rightarrow p_1^*\Delta \times p_2^*\Delta, \tag{7}$$

where $\text{Dec } \Delta$ denotes the cosimplicial bisimplicial set $[m] \mapsto \text{Dec } \Delta[m]$. This relies on (6) and the fact that Dec preserves products, together with the observation that the map (5) factors as

$$\text{Dec } \Delta \rightarrow \text{Dec}(\Delta \times \Delta)\delta \rightarrow (p_1^* \Delta \times p_2^* \Delta)\delta$$

Here the map $\Delta \rightarrow (\Delta \times \Delta)\delta$ is the canonical map inducing the counit $dd_* \rightarrow 1$ of the adjunction $d \dashv d_*$. The map (7) in turn factors as

$$\text{Dec } \Delta \times \text{Dec } \Delta \xrightarrow{1 \times \epsilon_c} \text{Dec } \Delta \otimes_2 \Delta \xrightarrow{\epsilon_r \times 1} p_1^* \Delta \times p_2^* \Delta,$$

where $\text{Dec } \Delta \otimes_2 \Delta$ denotes the bicosimplicial bisimplicial set which in bidegree (m, n) is $\text{Dec } \Delta[m] \otimes_2 \Delta[n]$. Applying the functor $\mathbf{SS}(-, X): cc\mathbf{SS} \rightarrow \mathbf{SS}$ we get a pair of maps of bisimplicial sets

$$(1 \times \epsilon_c)^*: \mathbf{SS}(\text{Dec } \Delta \otimes_2 \Delta, X) \rightarrow d_* TX \quad (8)$$

and

$$(\epsilon_r \times 1)^*: X \rightarrow \mathbf{SS}(\text{Dec } \Delta \otimes_2 \Delta, X). \quad (9)$$

We will prove that the first map is a row-wise weak equivalence and that the second map is a column-wise weak equivalence. Since d sends level-wise weak equivalences to weak equivalences, this is enough to prove that the map $dX \rightarrow dd_* TX$ is a weak equivalence.

Lemma 12. *The map (8) is a row-wise weak equivalence.*

Proof. The induced map on the n -th row induced by (8) is obtained by applying the functor $\mathbf{SS}(-, X)$ to the map of cosimplicial bisimplicial sets

$$\text{Dec } \Delta \times \text{Dec } \Delta[n] \rightarrow \text{Dec } \Delta \times p_2^* \Delta[n].$$

Since \mathbf{SS} is cartesian closed (it is a presheaf category), we see that the induced map on rows is given by the map of simplicial sets

$$\epsilon_c^*: \mathbf{SS}(p_2^* \Delta[n], X^{\text{Dec } \Delta}) \rightarrow \mathbf{SS}(\text{Dec } \Delta[n], X^{\text{Dec } \Delta}), \quad (10)$$

where $X^{\text{Dec } \Delta}$ denotes the simplicial object in \mathbf{SS} whose bisimplicial set of p -simplices is given by the exponential

$$X^{\text{Dec } \Delta[p]}$$

in \mathbf{SS} . Note that for any simplicial set K , there is a bijection

$$\mathbf{S}(K, \mathbf{SS}(\text{Dec } \Delta[n], X^{\text{Dec } \Delta})) = \mathbf{SS}(\text{Dec } \Delta[n], X^{\text{Dec } K}),$$

which is natural in K . Thus there is an isomorphism

$$\mathbf{SS}(\text{Dec } \Delta[n], X^{\text{Dec } \Delta})^{\Delta[1]} = \mathbf{SS}(\text{Dec } \Delta[n] \times \text{Dec } \Delta[1], X^{\text{Dec } \Delta}),$$

where $\mathbf{SS}(\text{Dec } \Delta[n], X^{\text{Dec } \Delta})^{\Delta[1]}$ denotes the simplicial path space of $\mathbf{SS}(\text{Dec } \Delta[n], X^{\text{Dec } \Delta})$. We will use these remarks to prove that the map (10) is a simplicial homotopy equivalence.

Note that the map $\sigma_c: p_2^* \Delta[n] \rightarrow \text{Dec} \Delta[n]$ induces a left inverse σ_c^* of the map ϵ_c^* . Therefore, we need to show that there is a simplicial homotopy

$$(\sigma_c \epsilon_c)^* \simeq \text{id}.$$

By the previous description of the simplicial path space of $\mathbf{SS}(\text{Dec} \Delta[n], X^{\text{Dec} \Delta})$, to find such a simplicial homotopy it suffices to find a map

$$\text{Dec} \Delta[n] \times \text{Dec} \Delta[1] \rightarrow \text{Dec} \Delta[n], \quad (11)$$

in \mathbf{SS} making the obvious diagram commute. A map as in (11) is given by the following composite:

$$\text{Dec} \Delta[n] \times \text{Dec} \Delta[1] \xrightarrow{1 \times \epsilon_r} \text{Dec} \Delta[n] \otimes_1 \Delta[1] \xrightarrow{k} \text{Dec} \Delta[n], \quad (12)$$

where $k: \text{Dec} \Delta[n] \otimes_1 \Delta[1] \rightarrow \text{Dec} \Delta[n]$ is the simplicial homotopy between $\sigma_c \epsilon_c$ and id of Lemma 9. It is easy to check that this map satisfies the required commutativity condition. \square

Lemma 13. *The map (9) is a column-wise weak equivalence.*

Proof. The map induced on the m -th column by (9) is obtained by applying the functor $\mathbf{SS}(-, X)$ to the map of cosimplicial bisimplicial sets

$$\epsilon_r \times 1: \text{Dec} \Delta[m] \times p_2^* \Delta \rightarrow p_1^* \Delta[m] \times p_2^* \Delta.$$

Again, since \mathbf{SS} is cartesian closed, we see that the induced map on columns is given by

$$\mathbf{SS}(p_1^* \Delta[m], X^{p_2^* \Delta}) \rightarrow \mathbf{SS}(\text{Dec} \Delta[m], X^{p_2^* \Delta}).$$

Our strategy again is to show that this map is a simplicial homotopy equivalence. Since σ_r is a right inverse of ϵ_r , the map $(\sigma_r \times 1)^*$ is a left inverse of $(\epsilon_r \times 1)^*$. Again, a little calculation shows that the simplicial path space of $\mathbf{SS}(\text{Dec} \Delta[m], X^{p_2^* \Delta})$ is given by

$$\mathbf{SS}(\text{Dec} \Delta[m] \times p_2^* \Delta[1], X^{p_2^* \Delta}).$$

Therefore, to find a simplicial homotopy $(\sigma_r \epsilon_r \times 1)^* \simeq \text{id}$, it suffices to find a bisimplicial map

$$\text{Dec} \Delta[m] \otimes_2 \Delta[1] \rightarrow \Delta[m].$$

making the obvious diagram commute. Such a map is given for example by the simplicial homotopy

$$h: \text{Dec} \Delta[m] \otimes_2 \Delta[1] \rightarrow \text{Dec} \Delta[m]$$

between $\sigma_r \epsilon_r$ and id of Lemma 9. This completes the proof of the lemma. \square

The proof of Proposition 11 is now complete, by the remark above. \square

In analogy with the fact that the map $dC \rightarrow \text{Tot } C$ of the generalized Eilenberg-Zilber Theorem for a chain complex C is a chain homotopy equivalence [6, 11], one may wonder whether the analogous map $dX \rightarrow TX$ of simplicial sets is a simplicial homotopy equivalence. In some cases this is known, for instance when X is the degree-wise nerve NG of a simplicial group G , as we will discuss in the next section. We suspect that the map $dX \rightarrow TX$ is not a simplicial homotopy equivalence for arbitrary X , however it seems a little difficult to construct a counter-example to support this. In this direction we can say the following however: there is no map $TX \rightarrow dX$ which is natural in X . For such a map would be induced in degree n by a map $\Delta[n, n] \rightarrow \text{Dec } \Delta[n]$, natural in $[n]$, which by adjointness would in turn be induced by a map $\Delta[2n + 1] \rightarrow \Delta[n]$, natural in $[n]$. However it is not hard to see that no such map can exist.

5 Kan's simplicial loop group construction revisited

The classifying complex $\overline{W}G$ of a simplicial group G was introduced in [10] (see Section 17 of that paper). We recall the definition.

Definition 14 (Eilenberg-Mac Lane [10]). Let G be a simplicial group. Then $\overline{W}G$ is the simplicial set with a single vertex, and whose set of n -simplices, $n \geq 1$, is given by

$$(\overline{W}G)_n = G_{n-1} \times G_{n-2} \times \cdots \times G_0.$$

The face and degeneracy maps of $\overline{W}G$ are given by the following formulas:

$$d_i(g_{n-1}, \dots, g_0) = \begin{cases} (g_{n-2}, \dots, g_0) & \text{if } i = 0, \\ (d_i(g_{n-1}), \dots, d_1(g_{n-i+1}), g_{n-i-1}d_0(g_{n-i}), g_{n-i-2}, \dots, g_0) & \text{if } 1 \leq i \leq n \end{cases}$$

and

$$s_i(g_{n-1}, \dots, g_0) = \begin{cases} (1, g_{n-1}, \dots, g_0) & \text{if } i = 0, \\ (s_{i-1}(g_{n-1}), \dots, s_0(g_{n-i}), 1, g_{n-i-1}, \dots, g_0) & \text{if } 1 \leq i \leq n. \end{cases}$$

The motivation for the above formula for $\overline{W}G$ is perhaps not so clear. We will show that there is a very natural ‘explanation’ for the above formula in terms of the décalage functors. For this, we first need some background on principal twisted cartesian products.

Recall that a *principal twisted cartesian product* (PTCP) with structure group G consists of a simplicial set P (the total space) and a simplicial set M (the base space) together with a map $\pi: P \rightarrow M$ and an action of G on P which is principal in the sense that the diagram

$$\begin{array}{ccc} P \times G & \longrightarrow & P \\ p_1 \downarrow & & \downarrow \pi \\ P & \xrightarrow{\pi} & M \end{array}$$

is a pullback, where p_1 denotes projection onto the first factor, the top arrow is the action of G on P , and π denotes the projection to the base. Moreover, $\pi: P \rightarrow M$ is required to have a *pseudo-cross section* (on the left), i.e. a family of sections σ_n of the maps $\pi_n: P_n \rightarrow M_n$ for all $n \geq 0$ such that $\sigma_{n+1}s_i = s_i\sigma_n$ for all $0 \leq i \leq n$ and $d_i\sigma_n = \sigma_{n-1}d_i$ for all $0 < i \leq n$.

The simplicial set $\overline{W}G$ is a classifying space for PTCPs with structure group G in the sense that there is a universal PTCP WG with base space $\overline{W}G$ with the property that every PTCP P on M with structure group G is induced by pullback from $WG \rightarrow \overline{W}G$ along a map $M \rightarrow \overline{W}G$, the classifying map of P .

In [7] Duskin explained how this classical notion of pseudo-cross section has a convenient reformulation in terms of Dec^0 . In this reformulation, σ is required to be a section of the induced map $\text{Dec}^0\pi: \text{Dec}^0P \rightarrow \text{Dec}^0M$ in the category $a_c\mathbf{S}$ of contractible augmented simplicial sets and coherent maps (see Section 2.2).

Since G acts principally on P , there is a canonical map of bisimplicial sets

$$\text{cosk}_0P \rightarrow NG,$$

where NG denotes the bisimplicial set which, when viewed as a (vertical) simplicial object in \mathbf{S} , has as its object of n -simplices the (horizontal) simplicial set NG_n , i.e. the nerve of the group G_n . Also here cosk_0P denotes the 0-coskeleton (or Čech nerve) of P , viewed as an object in \mathbf{S}/M . Therefore, cosk_0P has as its object of n -simplices the (horizontal) simplicial set $\check{C}(P_n)$ which is the Čech nerve of the map $\pi_n: P_n \rightarrow M_n$. In degree n the canonical map $\text{cosk}_0 \rightarrow NG$ is just the canonical map $\check{C}(P_n) \rightarrow NG_n$ arising from the principal action of G_n on P_n .

One of the advantages of this reformulation of the notion of PTCP is that it allows for a very simple and conceptual description of the classifying map of P (we find it hard to believe that this description was not known to Duskin). We have a commutative diagram

$$\begin{array}{ccc} \text{Dec}^0P & \longrightarrow & P \\ \downarrow & & \downarrow \\ \text{Dec}^0M & \longrightarrow & M. \end{array}$$

Composing the pseudo-cross section $s: \text{Dec}^0M \rightarrow \text{Dec}^0P$ with the map $\text{Dec}^0P \rightarrow P$ gives rise to a map $\text{Dec}^0M \rightarrow P$ over M which extends canonically to a simplicial map

$$\text{Dec } M \rightarrow \text{cosk}_0P$$

between simplicial objects in \mathbf{S}/M . Here Dec^0M is thought of as the vertical simplicial set of 0-simplices of the bisimplicial set $\text{Dec } M$. We can compose this with the canonical map $\text{cosk}_0P \rightarrow NG$ to obtain a map $\text{Dec } M \rightarrow NG$. The adjoint of the map $\text{Dec } M \rightarrow NG$ is a map

$$M \rightarrow TNG$$

which serves as a classifying map for P . One can go further and show that there is a canonical PTCP with base space TNG from which P arises via pullback along the above map. The next result shows that TNG is *precisely* the classifying complex $\overline{W}G$.

Lemma 15 (Duskin). *The classifying complex functor \overline{W} factors as*

$$\overline{W} = TN,$$

so that $\overline{W}G = TNG$ for any simplicial group G .

This factorization of \overline{W} is due as far as we know to Duskin, who observed that this factorization persists when simplicial groups are replaced by simplicially enriched groupoids, i.e. the functor $\overline{W}: \mathbf{SGpd} \rightarrow \mathbf{S}$ introduced by Dwyer and Kan in [8] also factors as $\overline{W} = TN$ (this last observation also appears in the MSc thesis of Ehlers [9]).

Proof. This is an essentially straightforward computation, so we will just give a sketch of the details. To an n -simplex of $\overline{W}G$ consisting of a tuple

$$(g_{n-1}, g_{n-2}, \dots, g_0)$$

as above, we associate the element (x_0, x_1, \dots, x_n) of TNG , where $x_0 = 1$ and

$$x_i = (d_0^{i-1}(g_{n-1}), d_0^{i-2}(g_{n-2}), \dots, d_0(g_{n-i+1}), g_{n-i}) \in (NG_{n-i})_i$$

for $i \geq 1$. This sets up a bijection $(\overline{W}G)_n = (TNG)_n$ which respects face and degeneracy maps. \square

It is well known that $\overline{W}G$ is weakly equivalent to the simplicial set dNG , obtained by applying the diagonal functor to the degree-wise nerve NG of the simplicial group G . Of course this can be seen as an instance of Theorem 1 in light of the identification $\overline{W}G = TNG$, but there are easier proofs, see for example [12]. In fact, $\overline{W}G$ is simplicially homotopy equivalent to dNG , the point being that both $\overline{W}G$ and dNG are fibrant (a proof of the latter fact can be found in [18]). In [23] it is shown via explicit calculation that the map $f: dNG \rightarrow \overline{W}G$ defined by

$$f(h_1, \dots, h_n) = (d_0(h_1), \dots, d_0^n(h_n))$$

for $h_i \in G_n$ exhibits dNG as a deformation retract of $\overline{W}G$. There is a further relationship between dNG and $\overline{W}G$ (see [2]): after passing to geometric realizations there is an *isomorphism* of spaces $|\overline{W}G| = |dNG|$. It is not clear that this isomorphism is induced by a simplicial map however. It would be interesting to give a more conceptual proof of the isomorphism from [2].

There are several advantages of the description of \overline{W} in Lemma 15 over the traditional description. One such advantage of the present description is that it becomes manifestly clear that \overline{W} has a left adjoint since both of the functors N and T do.

Proposition 16. *A left adjoint for the functor $\overline{W} = TN$ is given by the functor*

$$G = \pi_1 R \text{Dec}: \mathbf{S} \rightarrow s\mathbf{Gp},$$

where $R: \mathbf{SS} \rightarrow s\mathbf{S}_0$ is the left adjoint of the inclusion $s\mathbf{S}_0 \subset \mathbf{SS}$. If X is a simplicial set, then the value of G on X is the simplicial group GX defined by

$$[n] \mapsto \pi_1(\text{Dec}_n X / X_{n+1}).$$

Proof. Observe that the functor R is induced by the left adjoint of the inclusion $\mathbf{S}_0 \subset \mathbf{S}$, i.e. the functor which sends a simplicial set X to the reduced simplicial set $X/\mathrm{sk}_0 X$. To describe RX for X a bisimplicial set whose n -th row is X_n , we let $\mathrm{sk}_0 X$ denote the bisimplicial set whose n -th row is $\mathrm{sk}_0 X_n$, i.e. the constant simplicial set $[m] \mapsto X_{0,n}$. Then $RX = X/\mathrm{sk}_0 X$ so that the n -th row of RX is $RX_n = X_n/X_{0,n}$. The proposition then follows from the fact that $\mathrm{sk}_0 \mathrm{Dec}_n X$ is the constant simplicial set X_{n+1} . \square

Recall that a simplicial group G is said to be a *loop group* for a simplicial set X if there is a PTCP P on X with structure group G such that P is weakly contractible. In [13] Kan showed that the left adjoint $G: \mathbf{S} \rightarrow s\mathbf{Gp}$ of the classifying complex functor \overline{W} had the property that $G(X)$ was a loop group for any reduced simplicial set X . We will shortly give a simplified proof of his theorem by exploiting the description of G given in Proposition 16 above. Before we do this however we need the following lemmas.

Lemma 17. *Suppose that X is a bisimplicial set whose first column is weakly contractible, i.e. the simplicial set $[n] \mapsto X_{0,n}$ is weakly contractible. Then $X \rightarrow RX$ is a column-wise weak equivalence.*

Proof. For every $m \geq 0$, the vertical simplicial set $(\mathrm{sk}_0 X)_m$ is weakly contractible and so $X_m \rightarrow X_m/(\mathrm{sk}_0 X)_m$ is a weak equivalence of vertical simplicial sets for every $m \geq 0$. \square

Lemma 18. *Let X be a CW complex whose path components are all contractible. Then X/X^0 is a $K(\pi, 1)$, where X^0 denotes the set of vertices of X .*

Proof. X/X^0 can be written as a wedge

$$\bigvee_{\alpha \in \pi_0(X)} X_\alpha/X_\alpha^0,$$

where X_α denote the path components of X . Therefore without loss of generality we can assume that X is a path connected, pointed CW complex. We then have to show that X/X^0 is a $K(\pi, 1)$. Choose a strong deformation retraction of X onto a maximal tree T in the 1-skeleton X^1 of X (see for example I Theorem 5.9 of [26]). Then T/X^0 is a deformation retract of X/X^0 and so X/X^0 is a wedge of circles, from which the result follows. \square

Corollary 19. *For any simplicial set X , $\mathrm{Dec}_n X/X_{n+1}$ has the weak homotopy type of a $K(\pi, 1)$.*

Proof. Since $\mathrm{Dec}_n X = \mathrm{Dec}_0 \mathrm{Dec}_{n-1} X$, it is enough to prove this for $\mathrm{Dec}_0 X/X_1$. However this follows immediately from the Lemma since $\mathrm{Dec}_0 X$ deformation retracts onto X_0 (see Lemma 5). \square

With a little extra effort one can use this corollary to construct an explicit isomorphism between GX and the simplicial group described by Kan in [13], however we will not do this here.

We can now give a simple proof of Kan's result from [13] that $X \rightarrow \overline{W}GX$ is a weak equivalence when X is reduced. We will need the following property of the total simplicial set functor T : as observed in [3], since d sends level-wise weak equivalences of bisimplicial sets to weak equivalences of simplicial sets, Theorem 1 implies that this property is inherited by T . As an immediate consequence of this observation, Cegarra and Remedios prove the following:

Lemma 20 ([3]). *For any simplicial set X , the unit map $X \rightarrow T\text{Dec } X$ is a weak equivalence.*

We briefly review the proof of this result from [3].

Proof. Cegarra and Remedios observe that the composite of the unit $X \rightarrow T\text{Dec } X$ with the map $T\text{Dec } X \rightarrow Tp_1^*X$ is the identity on X , in light of the identification $Tp_1^*X = X$ of Lemma 10. Since T sends level-wise weak equivalences to weak equivalences it follows that $T\text{Dec } X \rightarrow X$ is a weak equivalence and hence the unit map is a weak equivalence. \square

We are now ready to prove that GX is a loop group for X whenever X is reduced.

Theorem 21 ([13]). *Let X be a reduced simplicial set. Then the unit map*

$$\eta: X \rightarrow \overline{W}GX$$

is a weak equivalence. Hence GX is a loop group for X .

Proof. The units of the adjunctions $\text{Dec} \dashv T$, $R \dashv U$, and $N_0 \dashv \pi_1$ give a factorization of η

$$X \rightarrow T\text{Dec } X \rightarrow TR\text{Dec } X \rightarrow TN\pi_1 R\text{Dec } X$$

in **S**. The map $X \rightarrow T\text{Dec } X$ is a weak equivalence by Lemma 20. The maps $T\text{Dec } X \rightarrow TR\text{Dec } X$ and $TR\text{Dec } X \rightarrow TN\pi_1 R\text{Dec } X$ are induced by the maps

$$\text{Dec } X \rightarrow R\text{Dec } X \text{ and } R\text{Dec } X \rightarrow N\pi_1 R\text{Dec } X$$

in **SS**. We will show that both of these maps are level-wise weak equivalences. The first map is a level-wise weak equivalence by Lemma 17, since $\text{sk}_0 \text{Dec } X = \text{Dec}^0 X$ and X is reduced. Corollary 19 shows that $R\text{Dec}_n X$ has the weak homotopy type of a $K(\pi, 1)$ and so the second map is also a level-wise weak equivalence. \square

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